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GRZEGORZ ŁUBCZONOK

Some remarks on connexions of the skew-symmetric tensor

Let M denote an n-dimensional differentiable manifold on which there is determined a nonsingular tensor field $a_{\lambda\mu}$. We search for linear connexions with respect to which the tensor field $a_{\lambda\mu}$ is covariantly constant, i.e.,

$$\nabla_\nu a_{\lambda\mu} = 0,$$

or

$$\Gamma^{\varrho}_{\lambda\nu} a_{\varrho\mu} + \Gamma^{\varrho}_{\mu\nu} a_{\lambda\varrho} = \partial_\nu a_{\lambda\mu}. \tag{1}$$

If the tensor field is symmetric and nonsingular, then there exists the affine connexion of Levi-Civita ([3], p. 131) fulfilling (1).

In § 1 we consider this question for the skew-symmetric and nonsingular tensor field $a_{\lambda\mu}$. The problem whether the object of linear connexion is a differential concomitant of $a_{\lambda\mu}$ is dealt with in § 2.

§ 1. We can write the solution of equation (1) in the form

$$\Gamma^{\varrho}_{\lambda\mu} = \overset{0}{\Gamma}{}^{\varrho}_{\lambda\mu} + X^{\varrho}_{\lambda\mu}, \tag{2}$$

where $\overset{0}{\Gamma}{}^{\varrho}_{\lambda\mu}$ is a particular solution of equation (1), $X^{\varrho}_{\lambda\mu}$ is the general solution of the equation

$$X^{\varrho}_{\lambda\nu} a_{\varrho\mu} + X^{\varrho}_{\mu\nu} a_{\lambda\varrho} = 0. \tag{3}$$

Now we are going to determine the general solution of (3). Equation (3) has the equivalent form

$$X^t_\nu a + a X^t_\nu = 0, \qquad \nu = 1, \ldots, n, \tag{4}$$

where $X_\nu = (X^{\varrho}_{\lambda})_\nu$, $a = (a_{\varrho\mu})$ and X^t_ν denotes the transpose of the matrix X_ν. Since the matrix a is skew-symmetric, we have

$$aX_\nu = (aX_\nu)^t, \qquad \nu = 1, \ldots, n.$$

Hence we have the following

LEMMA 1. The general solution of equation (3) is given by

$$X^{\varrho}_{\lambda\nu} = a^{\varrho\varkappa} S_{\varkappa\lambda\nu}, \tag{5}$$

where the tensor $S_{\varkappa\lambda\nu}$ fulfils $S_{[\varkappa\lambda]\nu} = 0$ and $a^{\varrho\varkappa}$ denotes the inverse tensor to $a_{\varrho\varkappa}$.

Now we shall describe a way to obtain particular solutions of equation (1). Let $\overline{\Gamma}^{\varrho}_{\lambda\nu}$ be any object of linear connexion. We shall try to find the connexion $\mathring{\Gamma}^{\varrho}_{\lambda\nu}$ in the form

$$\mathring{\Gamma}^{\varrho}_{\lambda\nu}=\overline{\Gamma}^{\varrho}_{\lambda\nu}+T^{\varrho}_{\lambda\nu}\,,$$

where $T^{\varrho}_{\lambda\nu}$ is any tensor. It follows from (1) that $T^{\varrho}_{\lambda\nu}$ must fulfil

$$T^{\varrho}_{\lambda\nu}\,a_{\varrho\mu}+T^{\varrho}_{\mu\nu}\,a_{\lambda\varrho}=\overline{\nabla}_{\nu}a_{\lambda\mu}\,, \tag{6}$$

where $\overline{\nabla}$ denotes the covariant derivative determined by $\overline{\Gamma}^{\varrho}_{\lambda\upsilon}$.

We see that the expresion $T^{\varrho}_{\lambda\nu}=-\frac{1}{2}a^{\varrho\tau}\overline{\nabla}_{\nu}a_{\lambda\tau}$ fulfils equation (6); in fact, we have

$$-\tfrac{1}{2}a^{\varrho\tau}\overline{\nabla}_{\nu}a_{\lambda\tau}a_{\varrho\mu}-\tfrac{1}{2}a^{\varrho\tau}\overline{\nabla}_{\nu}a_{\mu\tau}a_{\lambda\varrho}=\tfrac{1}{2}\delta^{\tau}_{\mu}\overline{\nabla}_{\nu}a_{\lambda\tau}-\tfrac{1}{2}\delta^{\tau}_{\lambda}\overline{\nabla}_{\nu}a_{\mu\tau}=\overline{\nabla}_{\nu}a_{\lambda\mu}\,.$$

Thus we have arrived at the following result.

THEOREM 1. *If on M^n there exists any linear connexion $\overline{\Gamma}^{\varrho}_{\lambda\nu}$, then exery connexion with respect to which the skew-symmetric and nonsingular tensor field $a_{\lambda\nu}$ is covariantly constant is given by the formula*

$$\Gamma^{\varrho}_{\lambda\nu}=\overline{\Gamma}^{\varrho}_{\lambda\nu}+a^{\varrho\varkappa}S_{\varkappa\lambda\nu}+\tfrac{1}{2}a^{\varrho\tau}\overline{\nabla}_{\nu}a_{\tau\lambda}$$

where $S_{\varkappa\lambda\nu}$ is an arbitary tensor fulfilling $S_{[\varkappa\lambda]\nu}=0$.

§ 2. Now we are going to investigate the question whether the object of linear connexion is a differential concomitant of rank p (cf. [2]) of the tensor $a_{\lambda\mu}$. The negative answer to this question is given in following

THEOREM 2. *If the tensor $a_{\lambda\mu}$ has a singular symmetric part, then the object of linear connexion is not a differential concomitant of $a_{\lambda\mu}$ of rank $p\leqslant 2$.*

The proof of the above theorem will be preceded by some preliminaries.

Let (G, X) denote an abstract object [1]. An object (G, Y) is a concomitant of (G, X) iff there exists a transformation $f:X\to Y$ which is onto and such that

$$f(g(x))=g(f(x)) \qquad \text{for} \qquad x\in X\,,\ g\in G\,.$$

The following remark (cf. [1]) is basic for our further considerations.

REMARK. If an object (G, Y) is a concomitant of (G, X), then

$$S_x\subset S_{f(x)} \qquad \text{for} \qquad x\in X\,,$$

where S_x and $S_{f(x)}$ denote the stability groups of the points x and $f(x)$, respectively.

Let L^n_2 denote the differential group of rank 2 ([2]) and let (L^n_2, M) be the geometric object with the space $M=\{(a_{\lambda\mu}, \partial_{\nu}a_{\lambda\mu})\}$. The action of L^n_2 on M is given by the formulas

$$\begin{aligned} a_{\lambda'\mu'}&=A^{\lambda}_{\lambda'}A^{\mu}_{\mu'}a_{\lambda\mu}\,,\\ \partial_{\nu'}a_{\lambda'\mu'}&=A^{\lambda}_{\nu'\mu'}A^{\mu}_{\mu'}a_{\lambda\mu}+A^{\mu}_{\nu'\mu'}A^{\lambda}_{\lambda'}a_{\lambda\mu}+A^{\nu}_{\nu'}A^{\lambda}_{\lambda'}A^{\mu}_{\mu'}\partial_{\nu}a_{\lambda\mu}\,. \end{aligned} \tag{7}$$

We denote by (L_2^n, Γ) the object of linear connexion. It has the following transformation formula

$$\Gamma^{\varrho'}_{\lambda'\nu'} = A^{\varrho'}_{\varrho} A^{\lambda}_{\lambda'} A^{\nu}_{\nu'} \Gamma^{\varrho}_{\lambda\nu} + A^{\varrho'}_{\varrho} A^{\varrho}_{\lambda'\nu'} . \tag{8}$$

Let $g_{\lambda\mu}$ and $s_{\lambda\mu}$ denote the symmetric and skew-symmetric part of $a_{\lambda\mu}$, respectively.

Proof of theorem 2. Without a loss of the generality we can assume that $\Gamma^{\varrho}_{\lambda\nu}$ is symmetric. Consider the elements of L_2^n of the form $(\delta^{\lambda}_{\lambda'}, A^{\lambda}_{\nu'\mu'})$, where $\delta^{\lambda}_{\lambda'}$ are KRONECKER's symbols. It follows from formulas (7) that the element $(\delta^{\lambda}_{\lambda'}, A^{\lambda}_{\nu'\mu'})$ belongs to the stability group of $(a_{\lambda\mu}, \partial_\nu a_{\lambda\mu})$ if

$$A^{\lambda}_{\nu'\lambda'} a_{\lambda\mu'} + A^{\mu}_{\nu'\mu'} a_{\lambda'\mu} = 0 \quad \text{for} \quad \nu', \lambda', \mu' = 1, \ldots, n . \tag{9}$$

In every transitive fiber of (L_n^2, M) there exists a point with the first components $a_{\lambda\mu} = \varepsilon_{\lambda\mu} + s_{\lambda\mu}$, where $\varepsilon_{\lambda\mu}$ is the canonical diagonal form of $g_{\lambda\mu}$:

$$(\varepsilon_{\lambda\mu}) = \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & 0\,\varepsilon_{\varkappa\varkappa} & & \\ & & & \ddots & \\ & & & & \varepsilon_{nn} \end{pmatrix}, \qquad \varepsilon^2_{\tau\tau} = 1, \ \tau = \varkappa, \ldots, n, \ \varkappa > 1 .$$

In particular, the numbers

$$\left\{ \begin{array}{l} A^{1}_{\nu'\lambda'} \ \text{arbitrary with} \ A^{1}_{\nu'\lambda'} = A^{1}_{\lambda'\nu'} , \\ \\ A^{\tau}_{\nu'\lambda'} = 0 \quad \text{for} \quad \tau > 1 , \end{array} \right. \tag{10}$$

are a solution of the equation

$$A^{\lambda}_{\nu'\lambda'} \varepsilon_{\lambda\mu'} + A^{\mu}_{\nu'\mu'} \varepsilon_{\lambda'\mu} = 0 , \qquad \nu', \mu', \lambda' = 1, \ldots, n .$$

Components (10) fulfil equation (9) if

$$A^{1}_{\nu'\lambda'} s_{1\mu'} + A^{1}_{\nu'\mu'} s_{\lambda'1} + \sum_{\tau=2}^{n} (A^{\tau}_{\nu'\lambda'} s_{\tau\mu'} + A^{\tau}_{\nu'\mu'} s_{\lambda'\tau}) = 0 .$$

From (10) it follows that

$$A^{1}_{\nu'\lambda'} s_{1\mu'} + A^{1}_{\nu'\mu'} s_{\lambda'1} = 0 , \qquad \text{hence} \qquad A^{1}_{\nu'\lambda'} s_{1\mu'} - A^{1}_{\nu'\mu'} s_{1\lambda'} = 0 ,$$

since $s_{\lambda\mu}$ is skew-symmetric.

Case 1. $s_{1\mu'} = 0$ for $\mu' = 1, \ldots, n$.

Then formulas (10) give a non-zero solution of (9).

Case 2. There is a μ' with $s_{1\mu'} \neq 0$.

Then formulas (10) with

$$A^{1}_{\nu'\lambda'} = s_{1\nu'} s_{1\lambda'} \quad \text{for} \quad \nu', \lambda' = 1, \ldots, n \tag{11}$$

give a non-zero solution of (9).

Thus at each case there exist non-zero elements $A^{\lambda}_{\nu'\mu'}$ such that $(\delta^{\lambda}_{\lambda'}, A^{\lambda}_{\lambda'\mu'})$ belongs to the stability group of the point M with the components $(\varepsilon_{\lambda\mu} + s_{\lambda\mu}, \partial_\nu(\varepsilon_{\lambda\mu} + s_{\lambda\mu}))$.

It follows from (8) that in the stability group of any point in Γ there do not exist elements $(\delta^{\lambda}_{\lambda'}, A^{\lambda}_{\nu'\mu'})$ with $A^{\lambda}_{\nu'\mu'} \neq 0$. We conclude that in every transitive fiber of (L^n_2, M) there exists a point whose stability group is not contained in the stability group of any point in Γ. This together with the Remark completes the proof.

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Grzegorz Łubczonok

PEWNE UWAGI O KONEKSJI DLA TENSORA SKOŚNIE SYMETRYCZNEGO

Streszczenie

W nocie zostały udowodnione dwa twierdzenia.

Twierdzenie 1. Jeżeli na rozmaitości M istnieje koneksja liniowa $\overline{\Gamma}^{\varrho}_{\lambda\nu}$, to każda koneksja o tej własności, że pewien tensor skośnie symetryczny i nieosobliwy $a_{\lambda\mu}$ jest względem niej kowariantnie stały, dana jest wzorem

$$\Gamma^{\varrho}_{\lambda\nu} = \overline{\Gamma}^{\varrho}_{\lambda\nu} + a^{\varrho\varkappa} S_{\varkappa\lambda\nu} + \tfrac{1}{2} a^{\varrho\tau} \overline{\nabla}_{\nu} a_{\tau\lambda}$$

gdzie $S_{\varkappa\lambda\mu}$ oznacza dowolny tensor spełniający warunek $S_{[\varkappa\lambda]\mu} = 0$.

Twierdzenie 2. Nie istnieje komitanta różniczkowa drugiego rzędu tensora kowariantnego $a_{\lambda\mu}$ o części symetrycznej osobliwej, która byłaby objektem koneksji liniowej.

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